Math 105 Assignment 4 Solutions

1. a) $$\int_{\pi/6}^{\pi/3} \frac{\cos(2x)\sin^5(x)}{\tan^2(x)}\,dx$$

$$= \int_{\pi/6}^{\pi/3} \frac{\cos(2x)\sin^5 x}{\frac{\sin^2 x}{\cos^2 x}}\,dx$$

$$= \int_{\pi/6}^{\pi/3} \cos(2x)\sin^3 x\cos^2 x\,dx$$

$$= \int_{\pi/6}^{\pi/3} (\cos^2 x - \sin^2 x)\sin^3 x\cos^2 x\,dx$$

$$= \int_{\pi/6}^{\pi/3} \sin^3 x\cos^4 x\,dx - \int_{\pi/6}^{\pi/3} \sin^5 x\cos^2 x\,dx$$

$$= \int_{\pi/6}^{\pi/3} \sin^2 x\cos^4 x\,\sin x\,dx - \int_{\pi/6}^{\pi/3} (\sin^2 x)^2\cos^2 x\cdot\sin x\,dx$$

$$= \int_{\pi/6}^{\pi/3} (1-\cos^2 x)\cos^4 x\,\sin x\,dx - \int_{\pi/6}^{\pi/3} (1-\cos^2 x)^2\cos^2 x\,\sin x\,dx$$

$u = \cos x$, $du = -\sin x\,dx$

$u(\pi/3) = \frac{1}{2}$, $u(\pi/6) = \frac{\sqrt{3}}{2}$

$$= \int_{\frac{\sqrt{3}}{2}}^{1/2} -(1-u^2)u^4\,du - \int_{\frac{\sqrt{3}}{2}}^{\frac{1}{2}} -(1-u^2)^2u^2\,du$$

$$= \int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} u^4 - u^6\,du - \int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} u^2 - 2u^4 + u^6\,du$$

$$= \int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} -u^2 + 3u^4 - 2u^6\,du$$

$$= \left[-\frac{u^3}{3} + \frac{3}{5}u^5 - \frac{2}{7}u^7 \right]_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}}$$

$$= \left[-\frac{1}{3}\left(\frac{\sqrt{3}}{2}\right)^3 + \frac{3}{5}\left(\frac{\sqrt{3}}{2}\right)^5 - \frac{2}{7}\left(\frac{\sqrt{3}}{2}\right)^7 \right] - \left[-\frac{1}{3}\left(\frac{1}{2}\right)^3 + \frac{3}{5}\left(\frac{1}{2}\right)^5 - \frac{2}{7}\left(\frac{1}{2}\right)^7 \right]$$

$$= -\frac{1}{3}\left(\frac{\sqrt{3}}{2}\right)^3 + \frac{3}{5}\left(\frac{\sqrt{3}}{2}\right)^5 - \frac{2}{7}\left(\frac{\sqrt{3}}{2}\right)^7 + \frac{1}{3}\left(\frac{1}{2}\right)^3 - \frac{3}{5}\left(\frac{1}{2}\right)^5 + \frac{2}{7}\left(\frac{1}{2}\right)^7$$

↑
ew

b) $\displaystyle\int_0^{\pi/4} \tan^5(x)\,dx$

$$= \int_0^{\pi/4} \tan^4 x \tan x\,dx$$

$$= \int_0^{\pi/4} (\sec^2 x - 1)^2 \tan x\,dx \quad , \quad u = \sec x \quad , \quad du = \tan x \sec x\,dx$$

$$u(0) = 1 \quad , \quad u(\pi/4) = \sqrt{2}$$

$$= \int_1^{\sqrt{2}} (u^2-1)^2 \frac{\tan x\,du}{\tan x \sec x}$$

$$= \int_1^{\sqrt{2}} (u^2-1)^2 \frac{du}{u}$$

$$= \int_1^{\sqrt{2}} \frac{u^4 - 2u^2 + 1}{u}\,du$$

$$= \int_1^{\sqrt{2}} u^3 - 2u + \frac{1}{u}\,du$$

$$= \left[\frac{u^4}{4} - u^2 + \log|u| \right]_1^{\sqrt{2}}$$

$$= \left[\frac{(\sqrt{2})^4}{4} - (\sqrt{2})^2 + \log\sqrt{2} \right] - \left[\frac{1}{4} - 1 + \log 1 \right]$$

$$= 1 - 2 + \frac{1}{2}\log 2 - \frac{1}{4} + 1 = \frac{1}{2}\log 2 - \frac{1}{4}$$

c) $\displaystyle\int \frac{y}{(y^2+4y+8)^{3/2}}\, dy$

Since there is a quadratic in a square root, we need to use trig sub. So to get it into a form for trig sub, we complete the square.

$$y^2+4y+8 = y^2+4y+4-4+8$$
$$=(y+2)^2+4$$

So $\displaystyle\int \frac{y}{(y^2+4y+8)^{3/2}}\, dy$

$$= \int \frac{y}{((y+2)^2+4)^{3/2}}\, dy \quad , \text{let } x = y+2 \ , dx = dy$$
$$y = x-2$$

$$= \int \frac{x-2}{(x^2+4)^{3/2}}\, dx \quad , \ x = 2\tan\theta \ , \ dx = 2\sec^2\theta\, d\theta$$

$$= \int \frac{2\tan\theta - 2}{(4\tan^2\theta+4)^{3/2}}\, 2\sec^2\theta\, d\theta$$

$$= 4\int \frac{(\tan\theta - 1)\sec^2\theta\, d\theta}{(4\sec^2\theta)^{3/2}}$$

$$= 4\int \frac{(\tan\theta - 1)\sec^2\theta\, d\theta}{8\sec^3\theta}\, d\theta$$

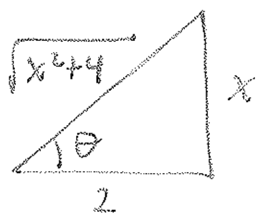
$$= \frac{1}{2}\int (\tan\theta - 1)\cos\theta\, d\theta$$

$$= \frac{1}{2}\int \sin\theta - \cos\theta\, d\theta$$

$$= -\frac{1}{2}\cos\theta - \frac{1}{2}\sin\theta + C$$

To find $\sin\theta$, $\cos\theta$, note

$$x = \tan\theta \Rightarrow \tan\theta = \frac{x}{2}$$



$$\rightarrow \sin\theta = \frac{x}{\sqrt{x^2+4}}$$

$$\cos\theta = \frac{2}{\sqrt{x^2+4}}$$

So $$\int \frac{y}{(y^2+4y+8)^{3/2}}\,dy = -\frac{1}{2}\,\frac{2}{\sqrt{x^2+4}} - \frac{1}{2}\,\frac{x}{\sqrt{x^2+4}} + C$$

$$= \frac{-2-x}{2\sqrt{x^2+4}} + C$$

$$= \frac{-2-(y+2)}{2\sqrt{(y+2)^2+4}} + C$$

$$= \frac{-4-y}{2\sqrt{y^2+4y+8}} + C$$

d) $$\int \frac{t^4+3t^3+6}{t^2+4t+3}\,dt$$

Since the degree of the numerator is higher than the degree of denominator, we long divide.

$$
\begin{array}{r|l}
 & t^2 - t + 1 \\
t^2+4t+3 & t^4 + 3t^3 + 0t^2 + 0t + 6 \\
 & \underline{t^4 + 4t^3 + 3t^2} \\
 & -t^3 - 3t^2 + 0t \\
 & \underline{-t^3 - 4t^2 - 3t} \\
 & t^2 + 3t + 6 \\
 & \underline{t^2 + 4t + 3} \\
 & -t + 3
\end{array}
$$

So $\displaystyle\int \frac{t^4+3t^3+6}{t^2+4t+3}\,dt = \int t^2 - t + 1 + \frac{3-t}{t^2+4t+3}\,dt$

$$= \frac{t^3}{3} - \frac{t^2}{2} + t + \int \frac{3-t}{(t+1)(t+3)}\,dt$$

We now apply partial fractions:

$$\frac{3-t}{(t+1)(t+3)} = \frac{A}{t+1} + \frac{B}{t+3}$$

$$= \frac{A(t+3) + B(t+1)}{(t+1)(t+3)}$$

So $\quad 3 - t = A(t+3) + B(t+1)$

$t = -1 \Rightarrow 3-(-1) = A(-1+3)$

$A = 2$

$t = -3 \Rightarrow 3-(-3) = B(-3+1)$

$B = -3$

So $\displaystyle\int \frac{3-t}{(t+1)(t+3)}\,dt = \int \frac{2}{t+1} - \frac{3}{t+3}\,dt$

$$= 2\log|t+1| - 3\log|t+3| + c$$

Thus $\displaystyle\int \frac{t^4+3t^2+6}{t^2+4t+3}\,dt = \frac{t^3}{3} - \frac{t^2}{2} + t + 2\log|1+t| - 3\log|3+t| + C$

e) $\displaystyle\int \frac{\cos x}{(\sin x - 1)^2(\sin x + 1)}\,dx$, let $u = \sin x$

$du = \cos x\,dx$

$$= \int \frac{1}{(u-1)^2(u+1)}\,du$$

Now let apply partial fractions.

$$\frac{1}{(u-1)^2(u+1)} = \frac{A}{u-1} + \frac{B}{(u-1)^2} + \frac{C}{u+1}$$

$$= \frac{A(u-1)(u+1) + B(u+1) + C(u-1)^2}{(u-1)^2(u+1)}$$

So $1 = A(u-1)(u+1) + B(u+1) + C(u-1)^2$

$= Au^2 - A + Bu + B + Cu^2 - 2Cu + C$

$= (A+C)u^2 + (B-2C)u + (B+C-A)$

$$\Rightarrow \begin{cases} A+C = 0 & (1) \\ B-2C = 0 & (2) \\ B+C-A = 1 & (3) \end{cases}$$

(1) $\Rightarrow$ $A = -C$

(2) $\Rightarrow$ $B = 2C$

(3) $\Rightarrow$ $2C + C - (-C) = 1$

So $4c = 1$ or $c = \frac{1}{4}$

Thus $A = -\frac{1}{4}$, $B = \frac{1}{2}$, $C = \frac{1}{4}$

$$\int \frac{1}{(u-1)^2(u+1)}\,du = \int -\frac{1}{4}\frac{1}{u-1} + \frac{1}{2}\frac{1}{(u-1)^2} + \frac{1}{4}\frac{1}{u+1}\,du$$

$$= -\frac{1}{4}\log|u-1| - \frac{1}{2}\frac{1}{(u-1)} + \frac{1}{4}\log|u+1| + C$$

So $$\int \frac{\cos x}{(\sin x-1)^2(\sin x+1)}\,dx = -\frac{1}{4}\log|\sin x - 1| - \frac{1}{2}\frac{1}{(\sin x-1)} + \frac{1}{4}\log|\sin x+1| + C$$

f) $$\int \frac{x}{x^2+6x+18}\,dx$$, let's complete the square.

$$x^2+6x+18 = x^2+6x+9-9+18$$
$$= (x+3)^2+9$$

$$\int \frac{x}{(x+3)^2+9}\,dx, \quad u = x+3, \ du = dx, \quad x = u-3$$

$$= \int \frac{u-3}{u^2+9}\,du$$

$$= \underbrace{\int \frac{u}{u^2+9}\,du}_{I_1} - \underbrace{\int \frac{3}{u^2+9}\,du}_{I_2}$$

$$I_1 = \int \frac{u}{u^2+9}\,du$$

There are 2 ways to do this, either trig sub or substitution.

Let's do substitution.

$t = u^2+9$, $dt = 2u\,du$ or $\frac{dt}{2} = u\,du$

$$\Rightarrow I_1 = \int \frac{1}{t}\,\frac{dt}{2}$$

$$= \frac{1}{2}\log|t| + C_1$$

$$= \frac{1}{2}\log|u^2+9| + C_1$$

$I_2 = \int \frac{3}{u^2+9}\,du$, let $u = 3\tan\theta$, $du = 3\sec^2\theta\,d\theta$

$$= \int \frac{3}{9\tan^2\theta+9}\cdot 3\sec^2\theta\,d\theta$$

$$= \int \frac{9\sec^2\theta}{9\sec^2\theta}\,d\theta$$

$$= \int d\theta$$

$$= \theta + C_2$$

$u = 3\tan\theta \Rightarrow \theta = \arctan\left(\frac{u}{3}\right)$

$I_2 = \arctan\left(\frac{u}{3}\right) + C$

So $\int \frac{x}{x^2+6x+18}\,dx = \frac{1}{2}\log|u^2+9| - \arctan\left(\frac{u}{3}\right) + C$

$= \frac{1}{2}\log|(x+3)^2+9| - \arctan\left(\frac{x+3}{3}\right) + C$

$= \frac{1}{2}\log|x^2+6x+18| - \arctan\left(\frac{x+3}{3}\right) + C$

2. a) $\int_0^a \frac{1}{\sqrt{a^2-x^2}}\,dx \quad , a>0$

Note $\frac{1}{\sqrt{a^2-x^2}}$ is not defined at $x=a$. So

$$\int_0^a \frac{1}{\sqrt{a^2-x^2}}\,dx = \lim_{t\to a^-}\int_0^t \frac{1}{\sqrt{a^2-x^2}}\,dx \quad , \quad \begin{aligned} x &= a\sin\theta \\ dx &= a\cos\theta\, d\theta \end{aligned}$$

$x=t \Rightarrow t = a\sin\theta$
$\theta = \arcsin\left(\frac{t}{a}\right)$

$x=0 \Rightarrow 0 = a\sin\theta$
$\theta = 0$

$$= \lim_{t\to a^-}\int_0^{\arcsin(\frac{t}{a})} \frac{1}{\sqrt{a^2-a^2\sin^2\theta}}\, a\cos\theta\, d\theta$$

$$= \lim_{t\to a^-}\int_0^{\arcsin(\frac{t}{a})} \frac{a\cos\theta}{a\cos\theta}\, d\theta$$

$$= \lim_{t\to a^-}\int_0^{\arcsin(\frac{t}{a})} d\theta$$

$$= \lim_{t\to a^-}\Big[\theta\Big]_0^{\arcsin(\frac{t}{a})}$$

$$= \lim_{t\to a^-} \arcsin\left(\frac{t}{a}\right)$$

$$= \arcsin(1)$$

$$= \frac{\pi}{2}$$

b) $\int_1^\infty \frac{\log x}{x^{3/2}}\,dx$

$= \lim_{b\to\infty} \int_1^b \frac{\log x}{x^{3/2}}\,dx$, $u = \log x$, $dv = \frac{1}{x^{3/2}}\,dx$

$du = \frac{1}{x}\,dx$ $\quad v = -\frac{2}{\sqrt{x}}$

$= \lim_{b\to\infty} -2\frac{\log x}{\sqrt{x}}\Big|_1^b - \int_1^b \frac{-2}{\sqrt{x}}\cdot\frac{1}{x}\,dx$

$= \lim_{b\to\infty} \left(-\frac{2\log b}{\sqrt{b}} + \frac{2\log 1}{\sqrt{1}}\right) + \int_1^b \frac{2}{x^{3/2}}\,dx$

$= \lim_{b\to\infty} \frac{-2\log b}{\sqrt{b}} + \left[\frac{-4}{\sqrt{x}}\right]_1^b$

$= \lim_{b\to\infty} -\frac{2\log b}{\sqrt{b}} - \frac{4}{\sqrt{b}} + 4$

$\lim_{b\to\infty} -\frac{4}{\sqrt{b}} = 0$

$\lim_{b\to\infty} \frac{-2\log b}{\sqrt{b}}$ is of form $\frac{\infty}{\infty}$

$= \lim_{b\to\infty} \frac{-2\frac{1}{b}}{\frac{1}{2\sqrt{b}}}$, by l'hopital's rule

$= \lim_{b\to\infty} \frac{-4}{\sqrt{b}}$

$= 0$

So $\int_1^\infty \frac{\log x}{x^{3/2}}\,dx = 4$

c) $\int_0^\infty x e^{-x^2} e^{\int_0^{x^2} e^{-t}\,dt}\,dx$

let $u(x) = \int_0^{x^2} e^{-t}\,dt$

so $\frac{du}{dx} = e^{-x^2}\cdot 2x$, by FTC I

$\frac{du}{2} = x e^{-x^2}\,dx$

Now we need to be careful of the bounds of integration.

We are interested between $x=0$ to "$x=\infty$". We need to find what happens to u at $x=0$ and as $x\to\infty$.

$\Rightarrow \int_{u(0)}^{"u(\infty)"} e^u \frac{du}{2}$.

$$"u(\infty)" = \lim_{x\to\infty} u(x)$$

$$= \lim_{x\to\infty} \int_0^{x^2} e^{-t}\,dt$$

$$= \lim_{x\to\infty} \left[-e^{-t}\right]_0^{x^2}$$

$$= \lim_{x\to\infty} 1 - e^{-x^2}$$

$$= 1$$

$$"u(0)" = \int_0^{0^2} e^{-t}\,dt$$

$$= 0$$

$$\Rightarrow \int_0^\infty x e^{-x^2} e^{\int_0^{x^2} e^{-t}\,dt}\,dx = \int_0^1 \frac{e^u}{2}\,du = \left[\frac{e^u}{2}\right]_0^1 = \frac{e}{2} - \frac{1}{2}$$

3. a) Let us find Δx, x_i.

$$a=0,\quad b=1,\quad n=6 \Rightarrow \Delta x = \frac{1-0}{6} = \frac{1}{6}$$

$$x_i = a + i\Delta x$$
$$= \frac{i}{6}$$

So $x_0 = 0$, $x_1 = \frac{1}{6}$, $x_2 = \frac{2}{6}$, $\ldots$, $x_6 = \frac{6}{6} = 1$

i) $$M(6) = \Delta x \sum_{i=1}^{6} f\left(\frac{x_{i-1}+x_i}{2}\right)$$
$$= \frac{1}{6}\sum_{i=1}^{6} f\left(\frac{\frac{i-1}{6}+\frac{i}{6}}{2}\right)$$
$$= \frac{1}{6}\sum_{i=1}^{6} f\left(\frac{i}{6} - \frac{1}{12}\right)$$
$$= \frac{1}{6}\left[e^{\left(\frac{1}{6}-\frac{1}{12}\right)^2} + e^{\left(\frac{2}{6}-\frac{1}{12}\right)^2} + e^{\left(\frac{3}{6}-\frac{1}{12}\right)^2} + e^{\left(\frac{4}{6}-\frac{1}{12}\right)^2} + e^{\left(\frac{5}{6}-\frac{1}{12}\right)^2} + e^{\left(\frac{6}{6}-\frac{1}{12}\right)^2}\right]$$
$$\approx 1.45641$$

ii) $$T(6) = \Delta x\left[\frac{1}{2}f(x_0) + f(x_1) + \cdots + f(x_5) + \frac{1}{2}f(x_6)\right]$$
$$= \frac{1}{6}\left[\frac{1}{2}e^{0^2} + e^{\left(\frac{1}{6}\right)^2} + e^{\left(\frac{2}{6}\right)^2} + e^{\left(\frac{3}{6}\right)^2} + e^{\left(\frac{4}{6}\right)^2} + e^{\left(\frac{5}{6}\right)^2} + \frac{1}{2}e^{1^2}\right]$$
$$\approx 1.47518$$

iii) $S(6) = \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + 4f(x_5) + f(x_6) \right]$

$$= \frac{\frac{1}{6}}{3}\left[e^{0^2} + 4e^{(\frac{1}{6})^2} + 2e^{(\frac{2}{6})^2} + 4e^{(\frac{3}{6})^2} + 2e^{(\frac{4}{6})^2} + 4e^{(\frac{5}{6})^2} + e^{1^2} \right]$$

$$\approx 1.46267$$

b) If we have $|f^{(4)}(x)| \le M$ for $x \in [a,b]$ then

$$\left| \int_a^b f(x)\,dx - S(n) \right| \le \frac{M(b-a)^5}{180 n^4}$$

I graciously told you that $|f^{(4)}(x)| \le 76e$, so $M = 76e$ works.

Thus,

$$\left| \int_0^1 e^{x^2}\,dx - S(6) \right| \le \frac{76e(1-0)^5}{180(6^4)}$$

$$= \frac{76e}{233280}$$

$$\approx 0.00088559\ldots$$

Thus $S(6)$ is accurate to 3 digits without even knowing what $\int_0^1 e^{x^2}\,dx$ is.

In fact $\int_0^1 e^{x^2}\,dx \approx 1.46265\ldots$

So it is actually accurate to 4 digits!

(FUN FACT!: $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$)

4. a) $\frac{dy}{dx} = xe^{x^2 - 2\log y}$

$= xe^{x^2} e^{-2\log y}$

$= xe^{x^2} e^{\log y^{-2}}$

$= xe^{x^2} y^{-2}$

$\Rightarrow \quad y^2 dy = xe^{x^2} dx$

$\int y^2 dy = \int xe^{x^2} dx \qquad , u = x^2, \; du = 2x\,dx$

$\frac{y^3}{3} = \int e^u \frac{du}{2}$

$= \frac{e^u}{2} + C_1$

$= \frac{e^{x^2}}{2} + C_1$

$\Rightarrow \quad y^3 = \frac{3e^{x^2}}{2} + C \qquad , C = 3C_1$

$y(x) = \sqrt[3]{\frac{3e^{x^2}}{2} + C}$

$y(0) = 0 \quad \Rightarrow \quad 0 = \frac{3}{2} e^{0^2} + C$

$C = -\frac{3}{2}$

So $\quad y(x) = \sqrt[3]{\frac{3}{2} e^{x^2} - \frac{3}{2}}$

b) $$\frac{dP}{dt} = kP(N-P)$$

$$\Rightarrow \frac{dP}{P(N-P)} = k\,dt$$

$$\int \frac{dP}{P(N-P)} = \int k\,dt$$

Let us first compute the partial fractions of $\frac{1}{P(N-P)}$.

$$\frac{1}{P(N-P)} = \frac{A}{P} + \frac{B}{N-P}$$

$$= A(N-P) + BP$$

$$\Rightarrow \quad 1 = A(N-P) + BP$$

$$P = 0 \Rightarrow 1 = A(N-0)$$

$$A = \frac{1}{N}$$

$$P = N \Rightarrow 1 = BN$$

$$B = \frac{1}{N}$$

$$\Rightarrow \frac{1}{P(N-P)} = \frac{1}{NP} + \frac{1}{N(N-P)}$$

So $$\int \frac{1}{P(N-P)}\,dP = \frac{1}{N}\int \frac{1}{P} + \frac{1}{N-P}\,dP$$

$$= \frac{1}{N}\left[\log|P| - \log|N-P|\right] + C$$

$$\int \frac{dP}{P(N-P)} = \int k\,dt$$

$$\Rightarrow \quad \frac{1}{N}\left[\log|P| - \log|N-P|\right] = kt + C_1$$

$$\log\left|\frac{P}{N-P}\right| = Nkt + C_2 \qquad , \quad C_2 = NC_1$$

$$\left|\frac{P}{N-P}\right| = e^{Nkt + C_2}$$

$$\text{or} \quad \frac{P}{N-P} = \pm e^{Nkt + C_2}$$

$$(\star) \qquad = A e^{Nkt} \qquad , \quad A = \pm e^{C_2}$$

Now we need to solve for P.

$$P = ANe^{Nkt} - Ae^{Nkt}P$$

$$P(Ae^{Nkt} + 1) = ANe^{Nkt}$$

$$P(t) = \frac{ANe^{Nkt}}{Ae^{Nkt} + 1}$$

$$= \frac{AN}{A + e^{-Nkt}}$$

Let us plug $P(0) = 1$ into $(\star)$ to solve for A.

$$\frac{1}{N-1} = A$$

$$\text{So} \quad P(t) = \frac{\frac{N}{N-1}}{\frac{1}{N-1} + e^{-Nkt}} = \frac{N}{1 + (N-1)e^{-Nkt}}$$